the progress of research is sure to be considerable. A student will have before him work whose style has never been surpassed, and demonstrations which are absolutely rigorous. In the latter respect Gauss' work seems to have left a lasting impression upon his mind.

I conclude by quoting the noble words from the British Association address:—

> "But in science sophistry is impossible; science knows no love of paradox; science has no skill to make the worse appear the better reason; science visits with a not long-deferred exposure all our fondness for preconceived opinions, all our partiality for views that we have ourselves maintained, and thus teaches the two best lessons that can well be taught—on the one hand the love of truth, and on the other sobriety and watchfulness in the use of the understanding."

P. A. MacMahon.

ABSTRACT GEOMETRY.

Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten in elementarer Form entwickelt. Von Guiseppe Veronese. (Leipzig: Teubner, 1894.)

MODERN speculations on the Foundations of Geometry have raised the question of the character of Geometry as a science, and the question has been answered in different ways. Some writers have held that our space-intuition is an absolute guarantee of the truth of geometrical axioms; others have treated Geometry as a science of observation and experience, whose results accordingly are liable to the same kind and degree of inexactness as any other Physical Science. If either of these answers were correct, the method of Geometry would seem to require revision. The method is to deduce the properties of figures by logical processes from definitions and a few propositions (Axioms) assumed in advance. But if space-intuition were a sufficient guarantee for the truth of the Axioms, it would seem to serve equally well for a guarantee of the truth of many of the Propositions, and there would appear to be no good reason for assuming as few as possible and deducing the rest. If, again, Geometry is to be purely a Natural Science, there would be simplicity in proving its propositions by the help of well-made constructions and good instruments of measurement. There seems to be room for a third view of Geometry as an abstract formal science to which the method always known as geometrical would be proper and natural. In such a view abstraction might be made of all space-intuition, and there would remain a body of logical truths in which the Axioms would occupy the place of Definitions or well-defined Hypotheses. The science would be at the same time founded upon intuition and independent of intuition. If its Definitions and Hypotheses are never in contradiction with themselves, or with each other, or with our space-intuition, then will its conclusions always be verified within the limits of exactness that belong to observation. It will be a formal science ready for practical applications.

The theory of Abstract Geometry in the sense just described is the subject-matter of Prof. Veronese's treatise. He lays down in his Preface the nature of Geometrical Axioms as the simplest truths of space-intuition; he describes the character of a system of Axioms in that they must be independent, as few as possible, and yet sufficient for the establishment of the properties of figures without tacit assumption of other axioms. No definition or axiom is satisfactory which contains any notion not previously cleared up, or anything to be afterwards deduced. Any geometrical figure regarded as existing in the space of intuition may be replaced by a well-defined mental object or "Form," in the sense of the word fully described in the Introduction. The geometrical axioms are replaced by hypotheses serving to discriminate among possible forms or possible formal relations. Intuition is taken as a guide to the choice of hypotheses. The distinction is drawn between Abstract Geometry and its practical applications, and it is pointed out that there may be axioms of great importance for the latter which are useless restrictions in the former: such axioms are that the space of intuition is the Euclidean space form, and that the space of intuition has three dimensions. For our author all conceivable space forms are in theory equally admissible, and the number of dimensions of space is unlimited. The straight line, the plane, space of three or n dimensions, are all regarded as existing in the General Space. His method is the method of Pure Geometry, and his work is free from any trace of axes, coordinates, and Algebraic processes. Apparently this method has not previously been applied to the discussion of space of more than three dimensions.

A reader who approached Prof. Veronese's book in the hope of finding a logical development in purely Geometrical form of the theories of the non-Euclidean Geometry would be disappointed, for the work is throughout subordinated to the Euclidean system; nor would the reader be better satisfied if he sought merely for the logical establishment of the Euclidean system, for it is throughout treated as a limit included in a more general possible system. It is well known that the Euclidean Geometry is the limiting form between the Hyperbolic and Elliptic Geometries, and this is the case whatever more particular character we attribute to either of these Geometries. Hyperbolic Geometries differ with the form chosen for the "Absolute," there are two Elliptic Geometries according as two straight lines have one or two common points. All these systems have Euclid's system as a limit. In the elements of an Abstract Geometry developed in an orderly way we shall be presented time after time with a choice of hypotheses. Our choice at any time will determine to some extent the space form of which we treat. Our series of hypotheses will limit us to a particular space form. If one of our hypotheses is the existence of straight lines, we shall come upon the Euclidean system or a non-Euclidean system having the Euclidean as its limit. We may state at once that Prof. Veronese's hypotheses lead him to a system which, in an absolute sense, is the so-called Spherical Geometry, as distinguished from the Elliptic Geometry proper. According to this system two straight lines cut in two "opposite" points, and the length between opposite points is constant. This, however, is only true in an "absolute sense," the length in question being actually infinite in comparison with any perceivable length treated as a unit. The doctrine of the "actually infinite" is that

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laid down in the Introduction. The artifice of using two units, the finite or Euclid's unit, and the infinitely great or Riemann's unit, is an essential part of the theory, and is referred to in the title of the book.

Let us now look a little more closely at the abstract development of Geometry as treated by Prof. Veronese. Such notions as "point" and "line" are suggested by simple intuitions; abstractly considered what are they? The point is simply the fundamental element of geometrical forms. It is an axiom that there are different points, but all points are identical. A straight line is a continuous point-system of one dimension, identical in the position of its parts, and determined by two of its points. Here it is to be observed that the straight line is not necessarily determined by *any* two of its points. It is an axiom that any point on the line and any point off the line determine only a single straight line. Hence if there are two or more points on a straight line by which it is not determined, any straight line through one goes through all. So far the Euclidean and non-Euclidean systems are not in any way discriminated. The choice of a system, excluding the Hyperbolic Geometry, is made by means of an hypothesis concerning different units. Let straight lines be drawn from a point, and any length in one of them chosen as a unit. On each of them there will be a "range of the scale" with that line as unit, and, as in the Introduction, there will be points outside the range of the scale. The points within the range of the scale form the finite domain about the point. The points at an actually infinite distance of the first order form the domain of the infinitely great of the first order. There can thus be a number of domains of infinitely great or infinitely small order of the space about a point. Suppose a point A taken on a straight line, and a point R outside it, and let the distance between them be chosen as a unit. Then if we join R to a point B on the line at a finite distance from A, the lines R B, A B are different relatively to the unit; if the distance A B is infinitely great in comparison with A R, they coincide relatively to the unit. The hypothesis which excludes the Hyperbolic Geometry is that two straight lines going from a point which in any domain are different relatively to the unit of that domain will not in any other domain coincide relatively to the unit of that domain, and it is proved that, on this hypothesis, two straight lines joining a point R to points at infinity in opposite directions on a line A B lie in the same straight line through R.

A point-system, defined as a straight line is defined, may be closed or open. In the former case starting from one point, and going through the system continuously in one direction, the point of starting will be ultimately arrived at; in the latter it will never be arrived at. If we assume the straight line open, and make the hypothesis just now described to exclude the Hyperbolic Geometry, we shall come to an absolute Euclidean Geometry. If we assume the straight line closed, but its entire length actually infinite in comparison with a perceivable unit, we shall come to a Spherical or Elliptic Geometry which coincides with the Euclidean in the domain of the perceivable unit about any point. This is the assumption chosen by Prof. Veronese. But there is still a choice open between the Spherical and the properly Elliptic Form. As mentioned above, the former is chosen by means of the hypothesis that a straight line contains pairs of points by which it is not determined. This hypothesis is adopted to avoid the kind of complication which occurs in the Elliptic Geometry, and which may be associated with the statement that the plane of the Elliptic Geometry is "unifacial" in the sense in which that word is used in Geometry of Position; but it is pointed out that for the purpose of obtaining a system including Euclid's as a limit, the hypothesis is a pure convention.

We have described the foundations of Prof. Veronese's system at considerable length, because it is by these that his system must be judged. For the subsequent developments it will be almost sufficient to say that they are clear and orderly, and, in places, very interesting. The construction of the plane by means of a pencil of rays meeting a straight line, leads to the essential properties of the plane and of plane figures. The like method by means of the "star" of rays from a point outside a plane to points on the plane, leads to the properties of figures in space of three dimensions. The word Star (*Stern*) is introduced in place of the older Sheaf of rays (*Strahlen-Bündel*). The construction of space of four dimensions is made by means of a star of rays from a point outside a space of three dimensions to the points within it, and so on for a space of any number of dimensions. Abstractly considered there cannot be in the nature of the case any restriction of Geometric Forms to space of three dimensions. All the forms—the straight line, plane, &c.—are treated first as Euclidean and afterwards as "complete" in the sense of the Spherical Geometry above described. The Euclidean forms first considered are regarded as the parts of the complete forms in the domain of the perceivable unit.

The use of more than one unit precludes the application to geometric magnitude of the axiom V of Archimedes; but there is another principle which has frequently been supposed to lie at the basis of Geometry with which our author also dispenses, we refer to the Principle of Superposition, or Motion without Deformation. He points out that, although this principle has been very extensively used as the test of equality, it yet involves in its statement the notion of equality, albeit in a limited form, and, as a test of equality, it is thus without meaning in an abstract sense. By placing the notion of equality of geometric magnitudes, or, as he says, identity of figures, on a different footing, he is enabled to prove the equality of congruent and symmetric figures, and to establish the idea of motion without deformation by means of continuous systems of identical figures.

The reader will see that the purpose of the book is not didactic, but the author hopes to produce a book adapted for learners, founded on the principles laid down, but limited to the Euclidean domain of a single unit. We shall look forward with much interest to its appearance. The indictment of Euclid is perhaps not yet complete, as almost every advance in Geometry throws light on some weakness in his logic, or defect in his method; but it is not too much to say that no well-reasoned didactic treatise on Elementary Geometry has yet appeared. In the meantime those who have studied the subject in the existing defective works will do well to clear their ideas by reading at least some parts of Prof. Veronese's.

A. E. H. L.

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